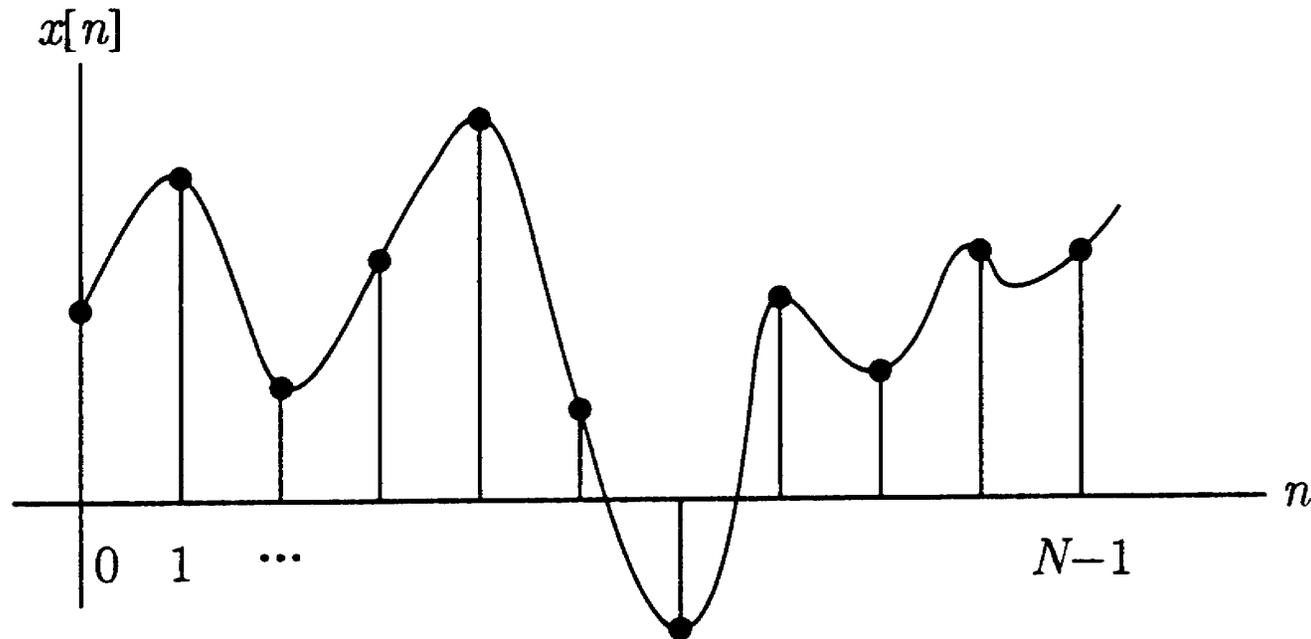


REPRESENTATION OF A RANDOM SIGNAL AS A RANDOM VECTOR



$$\mathbf{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

EXPECTATION FOR RANDOM VECTORS

DEFINITION

$$E\{\psi(\mathbf{x})\} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \psi(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}$$

$\psi(\mathbf{x})$: Any quantity (scalar, vector, matrix) depending on random vector \mathbf{x}

LINEARITY PROPERTY

$$E\{a\psi_1(\mathbf{x}) + b\psi_2(\mathbf{x})\} = a \cdot E\{\psi_1(\mathbf{x})\} + b \cdot E\{\psi_2(\mathbf{x})\}$$

MOMENTS OF A RANDOM VECTOR

MEAN VECTOR

$$\mathbf{m}_x = \mathcal{E}\{x\}$$

CORRELATION MATRIX

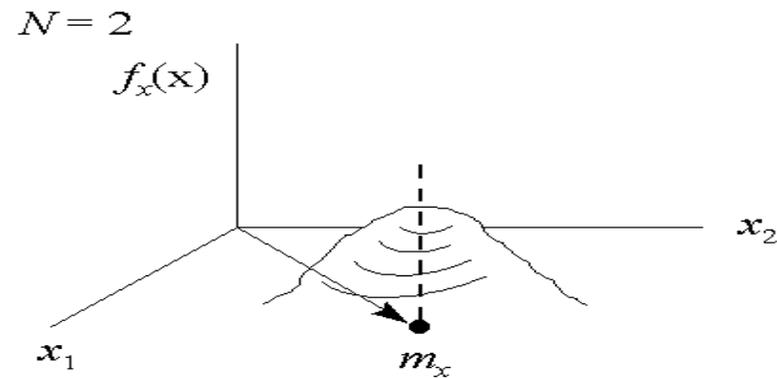
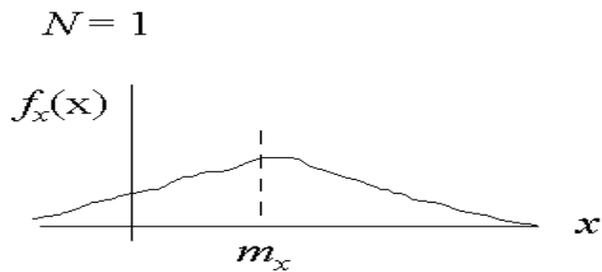
$$\mathbf{R}_x = \mathcal{E}\{xx^{*T}\}$$

COVARIANCE MATRIX

$$\mathbf{C}_x = \mathcal{E}\{(x - \mathbf{m}_x)(x - \mathbf{m}_x)^{*T}\}$$

MEAN VECTOR

$$\mathbf{m}_x = \mathcal{E}\{x\} = \begin{bmatrix} \mathcal{E}\{x_1\} \\ \mathcal{E}\{x_2\} \\ \vdots \\ \mathcal{E}\{x_N\} \end{bmatrix} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_N \end{bmatrix}$$



CORRELATION MATRIX

$$\begin{aligned}\mathbf{R}_x &= \mathcal{E} \left\{ \mathbf{x} \mathbf{x}^{*T} \right\} = \mathcal{E} \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \begin{bmatrix} x_1^* & x_2^* & \cdots & x_N^* \end{bmatrix} \right\} \\ &= \begin{bmatrix} \mathcal{E} \left\{ |x_1|^2 \right\} & \mathcal{E} \left\{ x_1 x_2^* \right\} & \cdots & \mathcal{E} \left\{ x_1 x_N^* \right\} \\ \mathcal{E} \left\{ x_2 x_1^* \right\} & \mathcal{E} \left\{ |x_2|^2 \right\} & \cdots & \mathcal{E} \left\{ x_2 x_N^* \right\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E} \left\{ x_N x_1^* \right\} & \mathcal{E} \left\{ x_N x_2^* \right\} & \cdots & \mathcal{E} \left\{ |x_N|^2 \right\} \end{bmatrix}\end{aligned}$$

COVARIANCE MATRIX

$$\mathbf{C}_x = \mathcal{E} \{ (\mathbf{x} - \mathbf{m}_x)(\mathbf{x} - \mathbf{m}_x)^{*T} \}$$

$$= \begin{bmatrix} \mathcal{E}\{|x_1 - m_1|^2\} & \mathcal{E}\{(x_1 - m_1)(x_2 - m_2)^*\} & \cdots & \mathcal{E}\{(x_1 - m_1)(x_N - m_N)^*\} \\ \mathcal{E}\{(x_2 - m_2)(x_1 - m_1)^*\} & \mathcal{E}\{|x_2 - m_2|^2\} & \cdots & \mathcal{E}\{(x_2 - m_2)(x_N - m_N)^*\} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}\{(x_N - m_N)(x_1 - m_1)^*\} & \mathcal{E}\{(x_N - m_N)(x_2 - m_2)^*\} & \cdots & \mathcal{E}\{|x_N - m_N|^2\} \end{bmatrix}$$

- Diagonal terms $\mathcal{E}\{|x_i - m_i|^2\}$ are *variances* of the vector components.

CORRELATION/COVARIANCE RELATION

$$\mathbf{C}_x = \mathbf{R}_x - \mathbf{m}_x \mathbf{m}_x^{*T}$$

Proof:

$$\begin{aligned} \mathcal{E} \left\{ (x - \mathbf{m}_x)(x - \mathbf{m}_x)^{*T} \right\} &= \mathcal{E} \left\{ xx^{*T} - x\mathbf{m}_x^{*T} - \mathbf{m}_x x^{*T} + \mathbf{m}_x \mathbf{m}_x^{*T} \right\} \\ &= \mathcal{E} \left\{ xx^{*T} \right\} - \mathcal{E} \left\{ x \right\} \mathbf{m}_x^{*T} - \mathbf{m}_x \mathcal{E} \left\{ x^{*T} \right\} + \mathbf{m}_x \mathbf{m}_x^{*T} \\ &= \mathcal{E} \left\{ xx^{*T} \right\} - \mathbf{m}_x \mathbf{m}_x^{*T} \end{aligned}$$

CORRELATION MATRIX PROPERTIES

1. Conjugate symmetry

$$\mathbf{R}_x = \mathbf{R}_x^{*T}$$

2. Positive semidefinite

$$\mathbf{a}^{*T} \mathbf{R}_x \mathbf{a} \geq 0$$

- These properties are *necessary and sufficient*.
- Identical properties hold for the covariance matrix.

PROOF OF PROPERTIES

Property 1:

$$\mathbf{R}_x^{*T} = \left(\mathcal{E} \{ \mathbf{x} \mathbf{x}^{*T} \} \right)^{*T} = \mathcal{E} \{ \mathbf{x} \mathbf{x}^{*T} \} = \mathbf{R}_x$$

Property 2:

$$\mathbf{a}^{*T} \mathbf{R}_x \mathbf{a} = \mathbf{a}^{*T} \mathcal{E} \{ \mathbf{x} \mathbf{x}^{*T} \} \mathbf{a} = \mathcal{E} \{ (\mathbf{a}^{*T} \mathbf{x})(\mathbf{x}^{*T} \mathbf{a}) \} = \mathcal{E} \{ |\mathbf{x}^{*T} \mathbf{a}|^2 \} \geq 0$$

MULTIVARIATE GAUSSIAN DENSITY

REAL RANDOM VECTOR

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}} |\mathbf{C}_{\mathbf{x}}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m}_{\mathbf{x}})^T \mathbf{C}_{\mathbf{x}}^{-1} (\mathbf{x}-\mathbf{m}_{\mathbf{x}})}$$

COMPLEX RANDOM VECTOR

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\pi^N |\mathbf{C}_{\mathbf{x}}|} e^{-(\mathbf{x}-\mathbf{m}_{\mathbf{x}})^*T \mathbf{C}_{\mathbf{x}}^{-1} (\mathbf{x}-\mathbf{m}_{\mathbf{x}})}$$

LINEAR TRANSFORMATIONS $y = Ax$

MEAN VECTOR

$$\mathcal{E}\{y\} = \mathcal{E}\{Ax\} = A\mathcal{E}\{x\} \quad \text{or ... } \boxed{m_y = Am_x}$$

CORRELATION MATRIX

$$\mathcal{E}\{yy^{*T}\} = \mathcal{E}\{(Ax)(Ax)^{*T}\} = A\mathcal{E}\{xx^{*T}\}A^{*T}$$

$$\text{or ... } \boxed{R_y = AR_xA^{*T}}$$

COVARIANCE MATRIX

$$\text{correspondingly ... } \boxed{C_y = AC_xA^{*T}}$$